

# The AGM Simple Pendulum

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## Abstract

We present a self-contained development of GAUSS' Arithmetic-Geometric Mean (AGM) and the work of A. E. INGHAM who obtained rigorous error bounds for the AGM's approximations to the period of a simple pendulum.

## 1 Introduction

One of the most celebrated problems in the classical dynamics of particles is the computation of the period of the simple pendulum. The nonlinear differential equation which models the pendulum's motion appears in numerous physical problems and the exact formula for the period of a single oscillation is given by a complete elliptic integral of the first kind. But, in 1834, JOSEPH LIOUVILLE [9] proved a justly famous theorem which implies that such an integral cannot be evaluated by any finite combination of elementary functions. Therefore, the calculation of the period must be carried out by suitable approximative formulas. This had been recognized long before, and in 1747, DANIEL BERNOULLI [12] published the first such approximation. Since then an enormous literature has arisen around the problem of finding a good approximation to the period and research continues unabated to this very day.

The authors of these approximations show great dexterity and ingenuity in their derivations and use a variety of techniques to obtain them. However, virtually NONE of them offers a rigorous error analysis. That is to say, inequalities on the upper bound for the error, which shows how *good* the approximation is, and on the lower bound for the error, which shows how *bad* the approximation is. (See THURSTON [13]). Most of the authors do include numerical studies of the accuracy of their approximations and some even include a few order-of-magnitude asymptotics. But those with rigorous error bounds are few and far between.

Recent interest has concentrated in GAUSS' Arithmetic-Geometric Mean (AGM) algorithm [5] because of its high rate of convergence. In 2008, CLAUDIO G. CARVALHAES and PATRICK SUPPES [2] published a very interesting and detailed presentation of the AGM and

its application to the approximation of the period. They also presented an elegant interpretation of the AGM recurrence formula as a method of *renormalizing* the pendulum in the sense that it replaces the original pendulum with another one with the same period, but longer length and smaller amplitude. This interpretation was already known to GREENHILL [6] in the late 1800's, but has been woefully neglected till recently. However, their paper, too, fails to offer any rigorous error analysis, although the numerical studies of the error are extremely interesting and merit study.

It is unfortunate that none of the authors cites the marvelous investigations of the great British number theorist A. E. INGHAM which L. A. PARS describes in his monumental 665-page standard work [11], which was published almost 50 years ago in 1965. INGHAM not only obtains the formulas of CARVALHAES and SUPPES but also obtains rigorous error estimates, both in excess and in defect. It is beyond question that Ingham's work deserves to be better known.

So, our paper is organized as follows. To make it as self-contained as possible, we develop *ab initio* the theory of the AGM including Gauss' original proof that it converges to the complete elliptic integral of the first kind. It is difficult to find this anywhere, today, since the clever method of D. J. NEWMAN [10] has now become fashionable. Then we apply the AGM to the case of the simple pendulum and we slightly alter the results and proofs of INGHAM so as to obtain a complete error analysis. Finally we compare our analytical error bounds with the numerical studies of CARVALHAES and SUPPES and show that they virtually coincide (as they should!). But, our analysis explains the *why* of their numerical results.

## 2 The AGM

We use the Stockholm lectures of VLADIMIR TKACHEV [14] in our treatment of the AGM.

GAUSS' only published account of his *arithmetic-geometric mean* (AGM) algorithm appears in a paper [5] on secular variations published in 1818. Yet, his unpublished papers [5] show that he was well aware of its major properties by 1799.

**Definition 2.1.** Let  $a \geq 0$  and  $b \geq 0$  be two numbers such that  $a \geq b \geq 0$  and define the numbers  $a_0$  and  $b_0$  by

$$a_0 := a, \quad b_0 := b; \tag{2.1}$$

then for  $n = 0, 1, 2, \dots$  define the sequences  $\{a_n\}$  and  $\{b_n\}$  by

$$a_{n+1} := \frac{a_n + b_n}{2}, \quad b_{n+1} := \sqrt{a_n b_n}. \tag{2.2}$$

Note that each  $a_{n+1}$  is the *arithmetic mean* of the previous  $a_n$  and  $b_n$ , while each  $b_{n+1}$  is the *geometric mean* of those same two numbers.

**Definition 2.2.** One says that the sequences  $\{a_n\}$  and  $\{b_n\}$  in (2.1) and (2.2) define the **arithmetic-geometric mean algorithm**, which we abbreviate as **AGM**.

Now we collect some of the basic properties of the AGM.

**Proposition 2.3.** *The following properties of the AGM are valid.*

1. *The  $a_n$ 's decrease, the  $b_n$ 's increase and every  $a_n$  is bigger than every  $b_m$ . More precisely,*

$$a_0 \geq a_1 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots \geq b_{n+1} \geq b_n \geq \cdots \geq b_1 \geq b_0. \quad (2.3)$$

- 2.

$$0 \leq a_n - b_n \leq \frac{a - b}{2^n}. \quad (2.4)$$

3. *The limits*

$$A := \lim_{n \rightarrow \infty} a_n \quad \text{and} \quad B := \lim_{n \rightarrow \infty} b_n \quad (2.5)$$

*both exist and they are equal,*

$$A = B. \quad (2.6)$$

*Proof.* Of (2.3):

Since the square of a real number is always non-negative, it follows that for  $n = 0, 1, 2, \dots$ ,  $(\sqrt{a_n} - \sqrt{b_n})^2 \geq 0$  and that there is strict inequality unless  $a_n = b_n$ , whence we conclude that the following inequality is valid,

$$\frac{a_n + b_n}{2} \geq \sqrt{a_n b_n}. \quad (2.7)$$

Of course, (2.7) is the famous *arithmetic-geometric mean inequality* for two numbers. Applying it to  $a_{n+1}$  and  $b_{n+1}$  we obtain

$$a_{n+1} \geq b_{n+1}. \quad (2.8)$$

Thus, from  $a_{n+1} \geq b_{n+1}$  and  $a_n \geq b_n$  we obtain

$$a_n \geq \frac{a_n + b_n}{2} =: a_{n+1} \geq b_{n+1} := \sqrt{a_n b_n} \geq b_n, \quad (2.9)$$

which is (2.3).

Of (2.4):

From  $b_{n+1} \geq b_n$  we conclude

$$a_{n+1} - b_{n+1} \leq a_{n+1} - b_n = \frac{a_n + b_n}{2} - b_n = \frac{a_n - b_n}{2}$$

and (2.4) follows by induction.

Of (2.5) and (2.6):

By (2.3) the sequence  $\{a_n\}$  decreases monotonically and is bounded from below by  $b_0$ , and

so  $A$  exists. By (2.3) the sequence  $\{b_n\}$  increases monotonically and is bounded from above by  $a_0$ , and so  $B$  exists.

Finally, letting  $n$  tend to infinity in (2.4) and using (2.5), we obtain

$$0 \leq A - B \leq 0$$

and by the “squeeze” theorem, we conclude  $A = B$ .  $\square$

Now the following definition makes sense.

**Definition 2.4.** We define the **arithmetic-geometric mean**,  $M(a, b) \equiv \mu$  of the numbers  $a$  and  $b$  to be the common limit

$$M(a, b) \equiv \mu := A := \lim_{n \rightarrow \infty} a_n \equiv B := \lim_{n \rightarrow \infty} b_n \quad (2.10)$$

of the AGM as applied to the numbers  $a$  and  $b$ .

**Proposition 2.5.** *The geometric mean  $b_n$  is a **closer** approximation to  $\mu$  than  $a_n$ ; more precisely*

$$0 < \frac{\mu - b_n}{a_n - \mu} < 1. \quad (2.11)$$

*Proof.* We observe

$$\mu < a_{n+1} = \frac{a_n + b_n}{2} \iff 2\mu < a_n + b_n \iff \mu - b_n < a_n - \mu.$$

Since  $0 < \mu - b_n < a_n - \mu$ , we can divide by  $a_n - \mu$  to complete the proof.  $\square$

### 3 Gauss’ theorem on elliptic integrals

The following theorem gives a hint of the depth of the mathematics involved in the AGM. It is the only theorem Gauss published on the algorithm and appears in the 1818 paper already cited [5]. But, it seems that he already had a proof in 1799 [3]:

**Theorem 3.1.** *Let  $a$  and  $b$  be positive real numbers. Then*

$$\boxed{\frac{1}{M(a, b)} = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}}}. \quad (3.1)$$

The integral (3.1) is a *complete elliptic integral of the first kind* and, as we have already seen [9], cannot be evaluated in finite terms with elementary functions. In the next section we will see its relationship to the simple pendulum.

Before we enter into the details of Gauss’ proof, we introduce some notation and separate out the fundamental technical step.

Let

$$I(a, b) := \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}}. \quad (3.2)$$

Then, we have to prove that

$$I(a, b) = I(a_1, b_1) = I(a_2, b_2) = I(a_3, b_3) = \dots \quad (3.3)$$

since we can then conclude that

$$I(a, b) = \lim_{n \rightarrow \infty} I(a_n, b_n) = I(\mu, \mu) = \frac{\pi}{2\mu} \quad (3.4)$$

which, after multiplying by  $\frac{2}{\pi}$ , is precisely (3.1).

In order to conclude

$$\lim_{n \rightarrow \infty} I(a_n, b_n) = I\left(\lim_{n \rightarrow \infty} a_n, \lim_{n \rightarrow \infty} b_n\right) = I(\mu, \mu) \quad (3.5)$$

we have to prove that *we can interchange the limit and the integral signs*. For this, it is sufficient to prove:

**Proposition 3.2.** *The sequence  $\left\{ \frac{1}{\sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi}} \right\}$ ,  $n = 0, 1, 2, \dots$ , converges **uniformly** to  $\frac{1}{\mu}$ .*

*Proof.* That means given any  $\epsilon > 0$  we must prove there exists positive number  $N(\epsilon)$ , which is *independent* of the variable  $\phi$ , such that the following implication is true:

$$n > N(\epsilon) \implies \left| \frac{1}{\sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi}} - \frac{1}{\mu} \right| < \epsilon. \quad (3.6)$$

However, the identity  $\cos^2 \phi + \sin^2 \phi = 1$  as well as the inequalities (2.3) and

$$b_n \leq \sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi} \leq a_n$$

show us that

$$-(a_n - b_n) = b_n - a_n < b_n - \mu < \sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi} - \mu < a_n - \mu < a_n - b_n,$$

that is,

$$\left| \sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi} - \mu \right| < a_n - b_n < \frac{a - b}{2^n} \quad (3.7)$$

where we applied (2.4) in the last inequality. Now,

$$\left| \frac{1}{\sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi}} - \frac{1}{\mu} \right| = \left| \frac{\sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi} - \mu}{\mu \cdot \sqrt{a_n^2 \cos^2 \phi + b_n^2 \sin^2 \phi}} \right| < \frac{a - b}{2^n} \frac{1}{b^2}$$

by (3.7) and (2.3). For the implication (3.6) to be true, it is sufficient that the following inequality be true:

$$\frac{a-b}{2^n} \frac{1}{b^2} < \epsilon \iff 2^n > \frac{a-b}{b^2 \epsilon} \iff n > \frac{\ln\left(\frac{a-b}{b^2 \epsilon}\right)}{\ln 2},$$

that is, the choice

$$N(\epsilon) := \frac{\ln\left(\frac{a-b}{b^2 \epsilon}\right)}{\ln 2} \quad (3.8)$$

proves the truth of the implication (3.6), and that, therefore, we can interchange the limit and integral signs in (3.5).  $\square$

Gauss' original proof is based on the following *change of variable in the integral*  $I(a, b)$ : we introduce a new variable,  $\phi'$  instead of  $\phi$  by the formula:

$$\boxed{\sin \phi =: \frac{2a \sin \phi'}{a + b + (a - b) \sin^2 \phi'}}. \quad (3.9)$$

**Proposition 3.3.** *Under the mapping (3.9) the interval  $0 \leq \phi' \leq \frac{\pi}{2}$  corresponds bijectively to the interval  $0 \leq \phi \leq \frac{\pi}{2}$ .*

*Proof.* Define the function

$$f(t) := \frac{2at}{a + b + (a - b)t^2}. \quad (3.10)$$

Then

$$f'(t) = 2a \frac{a + b - (a - b)t^2}{\{a + b + (a - b)t^2\}^2} \geq \frac{2ab}{\{a + b + (a - b)t^2\}^2} > 0, \quad (3.11)$$

which proves that  $f(t)$  is *increasing* on  $[0, 1]$ . Moreover,

$$f(0) = 0, \quad f(1) = 1,$$

which shows that  $f(t)$  maps  $[0, 1]$  bijectively onto itself. This completes the proof.  $\square$

*Proof of Gauss' theorem on elliptic integrals.* Gauss, himself [5], first states Proposition 3.3. Then he blithely asserts “*Evolutione autem rite facta, invenitur esse,*” which translates to “*After the development has been made correctly, it will be seen (that)...*”

$$\frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} = \frac{d\phi'}{\sqrt{a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi'}}. \quad (3.12)$$

(We have changed Gauss' notation: he writes  $m, n, m', n', T, T'$  in place of our  $a, b, a_1, b_1, \phi, \phi'$ , respectively.) This, of course, is the step

$$I(a, b) = I(a_1, b_1) \quad (3.13)$$

in the notation (3.2).

We will show how the development is “made correctly” (!).

*Claim 1:*

$$\cos \phi = \frac{2 \cos \phi' \sqrt{a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi'}}{a + b + (a - b) \sin^2 \phi'}. \quad (3.14)$$

*Proof.* By (3.9) and (2.2),

$$\begin{aligned} \cos^2 \phi &= 1 - \sin^2 \phi \\ &= 1 - \frac{4a^2 \sin^2 \phi'}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} \\ &= \frac{(a + b)^2 + 2(a^2 - b^2) \sin^2 \phi' + (a - b)^2 \sin^4 \phi' - 4a^2 \sin^2 \phi'}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} \\ &= \frac{4a_1^2 - 4(2a_1^2 - b_1^2) \sin^2 \phi' + 4(a_1^2 - b_1^2) \sin^4 \phi'}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} \\ &= \frac{4a_1^2 \cos^4 \phi' + 4b_1^2 \sin^2 \phi' \cos^2 \phi'}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} \end{aligned}$$

and factoring out  $4 \cos^2 \phi'$  and taking the square root of both sides gives us (3.14).  $\square$

*Claim 2:*

$$\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi} = a \frac{(a + b) + (a - b) \sin^2 \phi'}{(a + b) - (a - b) \sin^2 \phi'}. \quad (3.15)$$

*Proof.* By (3.14), (3.9) and (2.2), we obtain

$$\begin{aligned} a^2 \cos^2 \phi + b^2 \sin^2 \phi &= a^2 \left\{ \frac{2 \cos \phi' \sqrt{a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi'}}{a + b + (a - b) \sin^2 \phi'} \right\}^2 + \frac{4a^2 b^2 \sin^2 \phi'}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} \\ &= \frac{4a^2 \cos^2 \phi' (a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi') + 4a^2 b^2 \sin^2 \phi'}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} \\ &= 4a^2 \frac{a_1^2 (1 - \sin^2 \phi')^2 + b_1^2 \sin^2 \phi' (1 - \sin^2 \phi') + b^2 \sin^2 \phi'}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} \\ &= a^2 \frac{(a + b)^2 (1 - \sin^2 \phi')^2 + 4ab \sin^2 \phi' (1 - \sin^2 \phi') + (a - b)^2 \sin^4 \phi'}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} \\ &= a^2 \frac{(a + b)^2 - 2(a - b)(a + b) \sin^2 \phi' + 4b^2 \sin^2 \phi'}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} \\ &= \left\{ a \frac{(a + b) + (a - b) \sin^2 \phi'}{(a + b) - (a - b) \sin^2 \phi'} \right\}^2 \end{aligned}$$

and taking the square root of both sides gives us (3.15).  $\square$

Now we can complete the proof of Gauss' theorem. We take the differential of the left hand side of (3.9): we obtain

$$\cos \phi \, d\phi = \frac{2 \cos \phi' \sqrt{a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi'}}{a + b + (a - b) \sin^2 \phi'} d\phi',$$

where we applied (3.14).

Taking the differential of the right side of (3.9) we get

$$d \left\{ \frac{2a \sin \phi'}{a + b + (a - b) \sin^2 \phi'} \right\} = \frac{2a \cos \phi' \{(a + b) - (a - b) \sin^2 \phi'\}}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} d\phi'.$$

Equating the right hand side of the previous two equations we and using (3.15) we obtain

$$\begin{aligned} \frac{2 \cos \phi' \sqrt{a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi'}}{a + b + (a - b) \sin^2 \phi'} d\phi &= \frac{2a \cos \phi' \{(a + b) - (a - b) \sin^2 \phi'\}}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} d\phi' \\ \implies \frac{d\phi}{\sqrt{a^2 \cos^2 \phi + b^2 \sin^2 \phi}} &= \frac{a \{(a + b) - (a - b) \sin^2 \phi'\}}{\{(a + b) + (a - b) \sin^2 \phi'\}^2} d\phi' \cdot \frac{\frac{\{(a+b)-(a-b)\sin^2 \phi'\}^2}{a \{(a+b)-(a-b)\sin^2 \phi'\}}}{\sqrt{a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi'}} \\ &= \frac{d\phi'}{\sqrt{a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi'}}. \end{aligned}$$

This completes the proof of (3.12), therefore of (3.13), and therefore, of Gauss' theorem.  $\square$

## 4 The Simple Pendulum

First, we define the dynamical system. It is an idealization of a real pendulum.

**Definition 4.1.** The **simple pendulum** consists of a particle which is constrained to move without friction on the circumference of a vertical circle and which is acted upon only by gravity. We describe it mechanically as follows:

- a massless inextensible rigid rod has a point-mass attached to one end;
- the rod is suspended from a frictionless pivot;
- when the point-mass is given an initial push perpendicular to the rod, it will swing back and forth in one vertical plane and with a constant amplitude;
- there is no air resistance.

The following properties of the simple pendulum are readily available in numerous textbooks. For example, the standard work of PARS [11].

The *nonlinear differential equation* which models the motion of the simple pendulum is

$$\frac{d^2 \theta}{dt^2} + \frac{g}{l} \sin \theta = 0, \tag{4.1}$$



where  $g$  is the acceleration due to gravity,  $l$  is the length of the pendulum, and  $\theta(t)$  is the angular displacement, at time  $t$ , of the pendulum measured positively (counter-clockwise) from the vertical equilibrium position.

The *Period* of the pendulum,  $T$ , is the time taken by a double oscillation, to and fro, and is given by the following famous formula

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\alpha}{2} \sin^2 \phi}} \quad (4.2)$$

where  $\alpha$  is the maximum angular displacement of the pendulum.

The formula (4.2) shows that the period is proportional to a product of a function of the length,  $l$ , alone, and the maximum angular amplitude,  $\alpha$ , alone. That is, there is already a “separation of variables” in the formula for the period.

The integral in (4.2) is a *complete elliptic integral of the first kind* and, as we already noted, cannot be evaluated by any finite combination of elementary functions. So we must find suitable *approximative formulas* for  $K$ .

It is customary to write  $k := \sin \frac{\alpha}{2}$  so that the integral in (4.2) is

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}. \quad (4.3)$$

The quantity  $k$  is called the *modulus* of  $K$  and  $\frac{\alpha}{2}$  is called the *modular angle*. In our case, the modular angle is one-half of the maximum angular displacement of the pendulum, and we write (with an abuse of notation)  $K(k) \equiv K(\alpha)$ . The *complimentary modulus*,  $k' \geq 0$  is defined by  $k^2 + k'^2 = 1$  and the corresponding complete elliptic integral of the first kind is  $K(k') \equiv K'$ .

Expanding (4.3) by the binomial theorem and integrating term by term, we obtain

$$K(k) = \frac{\pi}{2} \left\{ 1 + \sum_{n=1}^{\infty} \left[ \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \right]^2 k^{2n} \right\}. \quad (4.4)$$

This gives us the fundamental theorem:

**Theorem 4.2.** *The period of a simple pendulum of length  $l$ , oscillating through an angle  $2\alpha$ , is equal to*

$$\begin{aligned} T &= 4\sqrt{\frac{l}{g}} \cdot K(k) \\ &= 2\pi\sqrt{\frac{l}{g}} \left\{ 1 + \left(\frac{1}{2}\right)^2 \left(\sin \frac{\alpha}{2}\right)^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 \left(\sin \frac{\alpha}{2}\right)^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 \left(\sin \frac{\alpha}{2}\right)^6 + \cdots \right\} \\ &= 2\pi\sqrt{\frac{l}{g}} \left\{ 1 + \frac{1}{16}\alpha^2 + \frac{11}{3072}\alpha^4 + \frac{173}{737280}\alpha^6 + \frac{22931}{1321205760}\alpha^8 + \cdots \right\}. \end{aligned}$$

□

The last formula in Theorem 4.2 comes from substituting the MACLAURIN expansion of  $\sin \frac{\alpha}{2}$  into the previous series, and rearranging in increasing powers of  $\alpha$ .

Thus, if a pendulum swinging through an angle of  $2\alpha$  makes  $N$  beats a day, and if  $\alpha$  is increased by  $\delta\alpha$ , then the formula  $N \cdot \frac{T}{2} = 86400$ , where 86400 is the number of seconds in a day, shows that *the pendulum will lose*

$$43200\sqrt{\frac{g}{l}}\left\{\frac{1}{K(\alpha)} - \frac{1}{K(\alpha + \delta\alpha)}\right\}$$

*beats a day.*

For example, a pendulum, which beats seconds when swinging through an angle of  $6^\circ$  will *lose* about  $11\frac{1}{2}$  seconds a day if made to swing through  $8^\circ$ , and about  $26\frac{1}{3}$  seconds a day if made to swing through  $10^\circ$  [6].

If we truncate the previous series expansions for the period, we obtain the following approximative formulas:

$$\begin{aligned} T &\approx 2\pi\sqrt{\frac{l}{g}} \\ &\approx 2\pi\sqrt{\frac{l}{g}}\left\{1 + \left(\frac{1}{2}\right)^2\left(\sin \frac{\alpha}{2}\right)^2\right\} \\ &\approx 2\pi\sqrt{\frac{l}{g}}\left\{1 + \frac{1}{16}\alpha^2\right\}. \end{aligned} \tag{4.5}$$

The first formula,

$$T \approx 2\pi\sqrt{\frac{l}{g}} \equiv T_0 \tag{4.6}$$

is the HUYGENS *formula* or the *small angle approximation* for the period. It does not contain  $\alpha$  and gives an approximation which is *independent* of the period. Indeed, it is the formula for the period of *simple harmonic motion*, (SHM), realized by a particle travelling in a circular path of with constant angular velocity  $\sqrt{\frac{g}{l}}$ .

Just how accurate is the Huygens formula? It seems worthwhile to cite the lower bound found by THURSTON [13] and the upper bound found by PARS [11].

**Corollary 4.3** (Pars–Thurston). *The Huygens small-angle approximation satisfies*

$$\frac{\frac{\alpha}{2}}{\sin \frac{\alpha}{2}} \leq \frac{T}{2\pi\sqrt{\frac{l}{g}}} \leq \sqrt{\frac{\alpha}{\sin \alpha}} \tag{4.7}$$

for  $0 \leq \alpha \leq \frac{\pi}{2}$ . □

As Thurston points out, if he were to use the Huygens formula to adjust the length of his grandfather clock, which has an amplitude of  $5^\circ$ , to beat seconds, the error bounds show that the clock would lose between 4 and 8 minutes per week.

These same bounds show that the Huygens formula is accurate to within 1% of the true period  $T$  for  $\alpha$  smaller than about  $14^\circ$ .

The second formula,

$$T \approx 2\pi \sqrt{\frac{l}{g}} \left\{ 1 + \left(\frac{1}{2}\right)^2 \left(\sin \frac{\alpha}{2}\right)^2 \right\} \quad (4.8)$$

tells us that in the correction for the amplitude of a swing, the period must be increased by the fraction  $\frac{1}{4} \sin^2 \frac{\alpha}{2}$  of itself. Thus, if a pendulum swinging through an angle of  $2\alpha$  makes  $N$  beats a day, and if  $\alpha$  is increased by  $\delta\alpha$ , *the pendulum will lose approximately  $(\frac{N}{8} \cdot \sin \alpha \cdot \delta\alpha)$  beats per day* [1].

The last formula

$$T \approx 2\pi \sqrt{\frac{l}{g}} \left\{ 1 + \frac{1}{16} \alpha^2 \right\} \quad (4.9)$$

is due to DANIEL BERNOULLI and is, historically, the *first* published correction term [12] to the Huygens formula (4.6).

## 5 The AGM approximations to the period

The previous section shows that the problem of finding an approximative formula for the period  $T$  of the simple pendulum reduces to the problem of approximating the complete elliptic integral  $K(k) \equiv K(\alpha)$ .

If we take  $a := 1$  and  $b := k' = \cos \frac{\alpha}{2}$  in the formula (3.2) for Gauss' integral  $I(a, b)$ , we obtain

$$I\left(1, \cos \frac{\alpha}{2}\right) = K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\alpha}{2} \sin^2 \phi}} \equiv \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}. \quad (5.1)$$

Applying the AGM to this choice of  $a$  and  $b$  we obtain the following sequences:

$$\begin{array}{ll} a_0 = 1 & b_0 = \cos \frac{\alpha}{2} \\ a_1 = \frac{1}{2} \left( 1 + \cos \frac{\alpha}{2} \right) = \cos^2 \frac{\alpha}{4} & b_1 = \left( \cos \frac{\alpha}{2} \right)^{\frac{1}{2}} \\ a_2 = \frac{1}{4} \left\{ 1 + \left( \cos \frac{\alpha}{2} \right)^{\frac{1}{2}} \right\}^2 & b_2 = \cos \frac{\alpha}{4} \left( \cos \frac{\alpha}{2} \right)^{\frac{1}{4}} \\ \quad = \frac{1}{2} \left\{ \cos^2 \frac{\alpha}{4} + \left( \cos \frac{\alpha}{2} \right)^{\frac{1}{2}} \right\} & \\ a_3 = \frac{1}{4} \left\{ \cos^2 \frac{\alpha}{4} + \left( \cos \frac{\alpha}{2} \right)^{\frac{1}{2}} \right\}^2 & b_3 = \frac{1}{2} \left\{ 1 + \left( \cos \frac{\alpha}{2} \right)^{\frac{1}{2}} \right\} \left( \cos \frac{\alpha}{4} \right)^{\frac{1}{2}} \left( \cos \frac{\alpha}{2} \right)^{\frac{1}{8}} \\ \dots = \dots & \dots = \dots \end{array}$$

We can use either  $\frac{1}{a_n}$  or  $\frac{1}{b_n}$  as an approximation to  $\frac{1}{\mu}$ .

In order to discuss the accuracy of these approximations, we recall some definitions from numerical analysis. See HILDEBRAND [7]. Each digit of a number, except zero, which serves only to fix the position of the decimal point is called a *significant digit* or a *significant figure* of that number.

**Definition 5.1.** (a) If any approximation  $\overline{N}$  to a number  $N$  has the property that both  $\overline{N}$  and  $N$  round to the same set of significant figures, and if  $n$  is the LARGEST integer for which this statement is true, then  $\overline{N}$  is said to **approximate  $N$  to  $n$  significant digits**.

(b)

$$R(\overline{N}) \equiv \text{relative error} := \frac{\text{true value} - \text{approximate value}}{\text{true value}} \equiv \frac{E(\overline{N})}{N}, \quad (5.2)$$

where  $E \equiv E(\overline{N})$  is the **absolute error**.

The importance of the *relative* error is shown in the following result.

**Proposition 5.2.**  $\overline{N}$  approximates  $N$  to  $n$  significant digits if and only if

$$R(\overline{N}) < \frac{(\frac{1}{2})}{10^n}. \quad (5.3)$$

**Proposition 5.3.** If  $R \equiv R(\overline{N})$  and

$$\overline{R} \equiv \overline{R}(\overline{N}) := \frac{E(\overline{N})}{\overline{N}}, \quad (5.4)$$

then

$$\overline{R} = \frac{R}{1 - R} \quad \text{and} \quad R = \frac{\overline{R}}{1 + \overline{R}}. \quad (5.5)$$

Now we are ready to present INGHAM's results.

**Theorem 5.4** (INGHAM). Let  $R_n$  be the relative error in the approximation  $\frac{1}{\mu} \approx \frac{1}{a_n}$  and  $r_n$  be the relative error in the approximation  $\frac{1}{\mu} \approx \frac{1}{b_n}$  taken positively. That is, let

$$\boxed{\left( \frac{1}{1 + r_n} \right) \cdot \frac{1}{b_n} := \frac{1}{\mu} =: \left( \frac{1}{1 - R_n} \right) \cdot \frac{1}{a_n}} \quad (5.6)$$

Then,

$$0 < r_n < R_n < \frac{a_n - b_n}{2a_{n+1}}. \quad (5.7)$$

*Proof.* If we divide the numerator and denominator of (2.11) by  $\mu$ , we see that the relative error,  $\bar{R}_n$ , in the approximation  $\mu \approx a_n$  is greater than the relative error,  $\bar{r}_n$ , in the approximation  $\mu \approx b_n$ . Substituting the formulas (5.5) into the inequality  $0 < \bar{r}_n < \bar{R}_n$ , some simple algebra leads us to the inequality

$$0 < r_n < R_n. \quad (5.8)$$

Moreover,

$$R_n = \frac{\frac{1}{\mu} - \frac{1}{a_n}}{\frac{1}{\mu}} = \frac{a_n - \mu}{a_n} < \frac{a_n - b_{n+1}}{a_n} = \frac{a_n - b_n}{a_n + b_{n+1}} < \frac{a_n - b_n}{2a_{n+1}} \quad (5.9)$$

where the second equality follows from

$$a_n^2 - b_{n+1}^2 = a_n(a_n - b_{n+1})$$

and the second inequality follows from

$$a_n + b_{n+1} > a_n + b_n = 2a_{n+1}. \quad \square$$

**Theorem 5.5** (INGHAM). *If  $T_0$  denotes the Huygens small-angle approximation to the true period,  $T$ , then, for  $0 < \alpha < \pi$ ,*

$$\boxed{\frac{T}{T_0} = \left\{ \frac{2}{1 + (\cos \frac{\alpha}{2})^{\frac{1}{2}}} \right\}^2 \cdot \left( \frac{1}{1 - R_2} \right) = \frac{1}{\cos \frac{\alpha}{4} (\cos \frac{\alpha}{2})^{\frac{1}{4}}} \cdot \left( \frac{1}{1 + r_2} \right)} \quad (5.10)$$

where

$$0 < r_2 < R_2 < \frac{1}{2^6 \cos \frac{\alpha}{2}} \left( \sin \frac{\alpha}{4} \tan \frac{\alpha}{4} \right)^4. \quad (5.11)$$

*Proof.* If we take  $n = 2$  in the error estimate (5.7), and using

$$8a_{n+1}(a_n - b_n) = (a_{n-1} - b_{n-1})^2, \quad (5.12)$$

we obtain

$$\begin{aligned} 0 < r_2 < R_2 < \frac{a_2 - b_2}{2a_3} &= \frac{1}{2a_3} \frac{(a_1 - b_1)^2}{8a_3} = \frac{1}{2a_3} \frac{1}{8a_3} \frac{(a_0 - b_0)^4}{(8a_2)^2} = \frac{\{\frac{1}{2}(a - b)\}^4}{2^6 a_3^2 a_2^2} \\ &= \frac{(\sin \frac{\alpha}{4})^8}{2^6 a_3^2 a_2^2} < \frac{(\sin \frac{\alpha}{4})^8}{2^6 b_2^4} = \frac{(\sin \frac{\alpha}{4})^8}{2^6 (\cos \frac{\alpha}{4})^4 \cos \frac{\alpha}{2}} \\ &= \frac{1}{2^6 \cos \frac{\alpha}{2}} \left( \sin \frac{\alpha}{4} \tan \frac{\alpha}{4} \right)^4. \quad \square \end{aligned}$$

**Corollary 5.6** (INGHAM). *If  $0 < \alpha \leq \frac{\pi}{2}$ , then*

$$0 < r_2 < R_2 < \frac{1}{70000} \quad (5.13)$$

and thus the approximation is correct to 5 significant figures in the worst case,  $\alpha = \frac{\pi}{2}$ .

*Proof.* If we take  $\alpha = \frac{\pi}{2}$  in (5.11) and note that

$$\sin \frac{\alpha}{4} = \sin \frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2}, \quad \cos \frac{\alpha}{4} = \cos \frac{\pi}{8} = \frac{\sqrt{2 + \sqrt{2}}}{2}, \quad (5.14)$$

and therefore

$$\frac{\cos^2 \frac{\alpha}{4}}{\sqrt{2} + 1} = \frac{\sin^2 \frac{\alpha}{4}}{\sqrt{2} - 1} = \frac{1}{2\sqrt{2}}, \quad (5.15)$$

we conclude

$$\begin{aligned} \frac{1}{2^6 \cos \frac{\alpha}{2}} \left( \sin \frac{\alpha}{4} \tan \frac{\alpha}{4} \right)^4 &= \frac{\sqrt{2}}{2^9 (\sqrt{2} + 1)^6} = \frac{\sqrt{2}}{2^9 (99 + 70\sqrt{2})} \\ &= \frac{99 - 70\sqrt{2}}{2^{8\frac{1}{2}}} \quad \{\text{since } (99)^2 - (70\sqrt{2})^2 = 1\} \\ &= \frac{1}{2^{8\frac{1}{2}} \cdot 70 \cdot 2.000050 \dots} \\ &< \frac{1}{2^{10} \cdot 70} = \frac{1}{71680} < \frac{1}{70000}. \end{aligned}$$

Finally, we have shown that  $R_2 < \frac{(\frac{1}{7})}{10^5} < \frac{(\frac{1}{2})}{10^5}$  and this means that the approximation is correct to 5 significant digits.  $\square$

The elegant calculations in this proof are due to Ingham [11]. If we apply the above computations to the case  $n = 3$ , we obtain the following results.

**Theorem 5.7** (INGHAM). *If  $T_0$  denotes the Huygens small-angle approximation to the true period,  $T$ , then, for  $0 < \alpha < \pi$ ,*

$$\boxed{\frac{T}{T_0} = \left\{ \frac{2}{\cos^2 \frac{\alpha}{4} + (\cos \frac{\alpha}{2})^{\frac{1}{2}}} \right\}^2 \cdot \left( \frac{1}{1 - R_3} \right)} \quad (5.16)$$

and

$$\boxed{\frac{T}{T_0} = \frac{2}{\{1 + (\cos \frac{\alpha}{2})^{\frac{1}{2}}\} (\cos \frac{\alpha}{4})^{\frac{1}{2}} (\cos \frac{\alpha}{2})^{\frac{1}{8}}} \cdot \left( \frac{1}{1 + r_3} \right)} \quad (5.17)$$

where

$$0 < r_3 < R_3 < \frac{1}{2^{14} \cos^2 \frac{\alpha}{2}} \left( \sin \frac{\alpha}{4} \tan \frac{\alpha}{4} \right)^8. \quad (5.18)$$

**Corollary 5.8** (INGHAM). *If  $0 < \alpha \leq \frac{\pi}{2}$ , then*

$$0 < r_3 < R_3 < \frac{1}{20000000000} \quad (5.19)$$

and thus the approximation is correct to 10 significant figures in the worst case,  $\alpha = \frac{\pi}{2}$ .

If we take  $\alpha = 179^\circ$  in (5.18), we obtain  $R_3 < 4.50 \dots \%$  which is in fair agreement with the machine calculation of [2] whose machine calculations showed  $R_3 \approx 1\%$ . In the next section we show how to bring it into much closer agreement.

## 6 Renormalization

Let  $k := \sin 75^\circ = \frac{\sqrt{6}+\sqrt{2}}{2}$  and let the complimentary modulus  $k' := \sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{2}$ . Let  $K$  y  $K'$  be the corresponding complete elliptic integrals of the first kind. In 1811, A.-M. LEGENDRE proved the following remarkable result [8]:

$$K = \sqrt{3} \cdot K'. \quad (6.1)$$

This equation implies that a pendulum with an amplitude of  $300^\circ$  and length  $l$  has *the same period* as a pendulum with an amplitude of  $60^\circ$  and a length  $3l$ .

Towards the end of the nineteenth century GREENHILL [6] proved: let the center of the circle of the pendulum's trajectory be  $O$ . Let  $B'B$  and  $b'b$  be two horizontal parallel chords of length  $\frac{l}{16}$  where  $B'B$  is above the center and  $b'b$  is below the center. Let  $k := \sin \frac{1}{4} \angle B' \hat{O} B$  and  $k' := \sin \frac{1}{4} \angle b' \hat{O} b$ . Then

$$K = \sqrt{7} \cdot K'. \quad (6.2)$$

This equation implies that a pendulum of length  $l$  and amplitude  $\angle B' \hat{O} B$  has *the same period* as a pendulum of length  $7l$  and amplitude  $\angle b' \hat{O} b$ .

That is, by suitably decreasing the amplitude and simultaneously increasing the length, one obtains a new pendulum with the same period. The smaller the amplitude, the more exact is each approximative formula we have developed. Thus, *if we perform this process of replacing a given pendulum with pendulums of longer lengths and smaller amplitudes, our approximations become better and better.*

This is an example of what today is called *renormalization* and it is a fundamental technique in the study of dynamical systems. The two examples, above, are taken from the deep and beautiful theory of *complex multiplication* of elliptic functions (curves), which is one of the most active branches of research on the frontiers of modern mathematics. Unfortunately, it lies outside the scope of our presentation (see [4] and [15]).

It turns out that the AGM furnishes us with another example of pendulum renormalization. Let us look at the equation

$$I(a, b) = I(a_1, b_1) \quad (6.3)$$

for the case  $a := 1$ ,  $b := \cos \frac{\alpha}{2}$  which gives us  $a_1 = \cos^2 \frac{\alpha}{4}$ ,  $b_1 = \cos \frac{\alpha}{2}$ . We note that

$$\frac{1}{\sqrt{a_1^2 \cos^2 \phi' + b_1^2 \sin^2 \phi'}} = \frac{1}{a_1} \frac{1}{\sqrt{1 - \frac{b_1^2}{a_1^2} \sin^2 \phi'}} = \frac{1}{\cos^2 \frac{\alpha}{4}} \frac{1}{\sqrt{1 - \tan^4 \frac{\alpha}{4} \sin^2 \phi'}}.$$

Returning to the formula for the period  $T$  of oscillation in an angle of  $4\alpha$ , we obtain the equation

$$\begin{aligned} T &= 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\alpha}{2} \sin^2 \phi}} = 4\sqrt{\frac{\left(\frac{l}{\cos^4 \frac{\alpha}{4}}\right)}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi'}{\sqrt{1 - (\tan^2 \frac{\alpha}{4})^2 \sin^2 \phi'}} \\ &=: 4\sqrt{\frac{l_1}{g}} \int_0^{\frac{\pi}{2}} \frac{d\phi'}{\sqrt{1 - \sin^2 \frac{\alpha_1}{2} \sin^2 \phi'}}. \end{aligned}$$

This equation can be stated as follows.

**Theorem 6.1.** *One iteration of the AGM transforms the pendulum of length  $l$  and maximum angular displacement  $\alpha$  into another pendulum with **the same period** but with new length*

$$\boxed{l_1 := \frac{l}{\cos^4 \frac{\alpha}{4}}} \quad (6.4)$$

which is **longer** than the original length  $l$ , and new maximum angular displacement

$$\boxed{\alpha_1 := 2 \cdot \arcsin\left(\tan^2 \frac{\alpha}{4}\right)} \quad (6.5)$$

which is **smaller** than the original. □

This theorem allows us to “explain” the results of the numerical investigations of the accuracy of the AGM which CARVALHAES and SUPPES presented in [2]. One uses the AGM to *renormalize* (reduce) the angular displacement  $\alpha$  so that the INGHAM estimates are applicable.

**Example 6.2.** CARVALHAES and SUPPES state that  $\frac{1}{a_2}$  approximates  $\frac{T}{T_0}$  to within 1% for  $0 \leq \alpha \leq 163.10^\circ$  while the INGHAM bound (5.11) gives  $0 \leq \alpha \leq 162.5^\circ$ . Again they report that the approximation has a relative error no bigger than  $\frac{1}{2^{52}}$  for  $0 \leq \alpha \leq 4.57^\circ$  while the INGHAM bound (5.11) gives  $0 \leq \alpha \leq 4.258^\circ$ .

**Example 6.3.** As another example, CARVALHAES and SUPPES state that  $\frac{1}{a_4}$  approximates  $\frac{T}{T_0}$  to within 1% for  $0 \leq \alpha \leq 179.99^\circ$ . The INGHAM bound (5.18) gives that  $\frac{1}{a_3}$  approximates  $\frac{T}{T_0}$  to within 1% for  $0 \leq \alpha \leq 177.98^\circ$ . But, one application of the AGM reduces  $\alpha = 179.99^\circ$  to  $\alpha = 177.85^\circ$  and now the INGHAM bound (5.18) shows that *three more* applications of the AGM give an approximation to within 1%, that is,  $\frac{1}{a_4}$  approximates  $\frac{T}{T_0}$  to within 1% for  $0 \leq \alpha \leq 179.99^\circ$ , in agreement with [2].



The remaining results in [2] can be “explained” similarly.

We cannot emphasize strongly enough the importance of rigorous upper and lower bounds for the absolute and relative errors in approximative formulas. Our analysis allows us to predict “*a priori*” the accuracy of a given approximative formula as well as to justify the resulting numerical studies. Moreover, INGHAM’s elegant and beautiful investigations give us practical tools to tailor our approximative formulas to the needs of the accuracy demanded.

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